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Müntz-Type Problems for Bernstein Polynomials

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We examine how many of the Bernstein basis functions $x^k(1-x)^{n-k}$, $k = 0, \dots, n$, can be omitted such that linear combinations of the remaining polynomials are still dense in the space of continuous functions.

1. INTRODUCTION

It is well-known that the Bernstein basis functions $x^k(1-x)^{n-k}$, $0 \le k \le n$, provide a convenient tool of approximation of continuous functions on $[0, 1]$. In this note, following a suggestion of Borwein, we consider the following Müntz-type problem: How many Bernstein basis functions can be omitted so that the approximation of continuous functions is still possible? Let $R_n \subset \{1, 2, \dots, n-1\}$ be an arbitrary set of integers ($n = 2, 3, \dots$), and consider the following subspace of polynomials of degree at most n:

$$\mathscr{P}(R_n) = \operatorname{span}\left\{x^k(1-x)^{n-k} : 0 \le k \le n,\ k \notin R_n\right\}.$$

(Note that for the density in $C[0,1]$, it is necessary to keep the first and last basis functions $(1-x)^n$ and x^n.) Furthermore, we shall say that $\mathscr{P}(R_n)$ approximates $C[0,1]$, i.e.,

$$\operatorname*{Lim}_{n\to\infty} \mathscr{P}(R_n) = C[0,1], \tag{1}$$

if for every $f \in C[0,1]$ there exist $p_n \in \mathscr{P}(R_n)$, $n = 2, 3, \dots,$ such that $\|f - p_n\| \to 0$ as $n \to \infty$. (In what follows, $\|\cdot\|$ will always mean supremum norm over the interval $[0, 1]$.)

This problem is somewhat different from the classical Müntz problem where approximation is required by a nested sequence of basis polynomials. Here, in general, $\mathscr{P}(R_n)$ and $\mathscr{P}(R_m)$ are different if $n \ne m$.

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Our aim in this paper is to investigate under what conditions on R_n the relation (1) holds. As a by-product, we shall also settle the problem when $C[0, 1]$ is replaced by

$$C^*[0,1] := \{f: f \in C[0,1], f(0) = f(1) = 0\}.$$

It will turn out that in these problems the "distance"

$$\varrho(R_n) := \min\{r, n - r: r \in R_n\}$$

from R_n to the endpoints of the interval $[0, n]$ plays an important role. (Since $R_n \subset \{1, 2, \ldots, n - 1\}$, we always have $1 \le \varrho(R_n) \le n/2$.) Another factor which comes naturally into play is $\#R_n$, the cardinality of R_n. Note that $\#R_n + 2\varrho(R_n) \le n + 1$ for every R_n. The problem outlined above possesses different solutions depending on whether $\varrho(R_n) = O(1)$, or $\varrho(R_n) \to \infty$ as $n \to \infty$. (For simplicity of writing, we do not consider the case when $\limsup_{n\to\infty} \varrho(R_n) = \infty$, since then for corresponding subsequences the corresponding statements hold.)

2. The Space $C[0, 1]$

Theorem 1. *Let $1 \le \varrho \le n/2$ be a fixed integer, and let $\{r_n\}$ $(r_n \le n + 1 - 2\varrho)$ be an increasing sequence of integers. Then in order that for every $R_n \subset \{1, 2, \ldots, n - 1\}$ with $\#R_n = r_n$ and $\varrho(R_n) = \varrho$ $(n = 1, 2, \ldots)$ the relation* (1) *hold it is necessary and sufficient that $r_n = o(\sqrt{n})$.*

Theorem 2. *Let $\{r_n\}, \{\varrho_n\} (r_n + 2\varrho_n \le n + 1)$ be increasing sequences of integers and assume $\varrho_n \to \infty$ as $n \to \infty$. Then in order that for every $R_n \subset \{1, 2, \ldots, n - 1\}$ with*

$$\#R_n = r_n \quad \text{and} \quad \varrho(R_n) = \varrho_n \quad (n = 1, 2, \ldots), \tag{2}$$

relation (1) *hold, it is sufficient that*

$$\limsup_{n\to\infty} \frac{r_n^2}{n\varrho_n} < \frac{1}{2^{15}e^2}, \tag{3}$$

and necessary that

$$\limsup_{n\to\infty} \frac{r_n^2}{n\varrho_n} < 53. \tag{4}$$

Summarizing the above statements we can say that when $\varrho(R_n) = O(1)$ then the condition $\#R_n = o(\sqrt{n})$ is necessary and sufficient for (1) to hold.

Furthermore, if $\varrho(R_n) \to \infty$ as $n \to \infty$, then $\#R_n = O(\sqrt{n\varrho_n})$ provides the necessary and sufficient condition for (1). The second result also shows that in choosing R_n we can drop more numbers from the "middle" than from the "ends" of the set $\{1, \ldots, n-1\}$.

We shall need some well-known facts concerning the so-called incomplete polynomials. Polynomials of the form $p_n(x) = \sum_{k=s}^{n} a_k x^k$ where $s \geq [n\theta]$, are called θ-incomplete at 0 $(0 < \theta < 1)$. It is known that if $|p_n(\xi)| = \|p_n\|$ for some $0 \leq \xi \leq 1$ and θ-incomplete polynomial p_n, then $\xi > \theta^2$. Furthermore, if g_k is a sequence of θ-incomplete polynomials with $\deg g_k \to \infty$ and $\|g_k\| \leq 1$ then $\lim_{k\to\infty} g_k = 0$ uniformly on compact subsets of $[0, \theta^2)$ (see the survey paper [4] of Lorentz).

We shall need the following:

LEMMA 1. *Let $0 < \theta < 1$, $n \in \mathbf{N}$, $m \geq (1-\theta)n$, and consider arbitrary distinct integers $0 < \lambda_j \leq n$, $1 \leq j \leq m$. Then for every θ_0, $\theta < \theta_0 < 1$, we have*

$$E_n := \min_{c_j} \max_{\theta_0^2 \leq x \leq 1} \left| 1 - \sum_{j=1}^{m} c_j x^{\lambda_j} \right| \leq \left(\frac{\theta + \theta_0}{2\theta_0} \right)^{(\theta_0 - \theta)n/2}. \tag{5}$$

Proof. With proper numbers c_j, $j = 1, \ldots, m$, and arbitrary $s > 0$, we have (cf. von Golitschek [1])

$$\begin{aligned} E_n &\leq \theta_0^{-2s} \max_{\theta_0^2 \leq x \leq 1} \left| x^s - \sum_{j=1}^{m} c_j x^{s+\lambda_j} \right| \leq \theta_0^{-2s} \max_{0 \leq x \leq 1} \left| x^s - \sum_{j=1}^{m} c_j x^{s+\lambda_j} \right| \\ &\leq \theta_0^{-2s} \prod_{j=1}^{m} \frac{\lambda_j}{\lambda_j + 2s} \leq \theta_0^{-2s} \prod_{j=n-m+1}^{n} \frac{j}{j+2s} \\ &\leq \theta_0^{-2s} \frac{(n-m+1)\cdots(n-m+2s)}{(n+1)\cdots(n+2s)} \leq \theta_0^{-2s} \left(\frac{n-m+2s}{n+2s} \right)^{2s}, \end{aligned}$$

whence (5) follows by setting $s = (\theta_0 - \theta)n/4$.

Remark. From Lemma 1 we can easily derive the well-known fact that any function continuous on $[\theta_0^2, 1]$ can be approximated by θ-incomplete polynomials if $\theta < \theta_0$ (see von Golitschek [2] and Saff and Varga [5]).

Since the proofs of sufficiency and necessity of Theorems 1 and 2 follow similar lines, it will be convenient to verify first the sufficiency and then the necessity of both statements.

Proof of Sufficiency in Theorems 1 *and* 2. Let R_n be a subset of $\{1, \ldots, n-1\}$ with (2). We start by approximating an $f \in C[0,1]$ via its

nth Bernstein polynomial

$$B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Now we need to approximate $B_n(f, x)$ by polynomials from $\mathscr{P}(R_n)$. Evidently, it will suffice to provide an approximation for

$$\tilde{B}_n(f, x) := \sum_{\varrho_n \le k \le n/2} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

(Namely, our considerations can be repeated with the substitution $x = 1 - y$.) With an a, $0 < a < 1/4$, to be determined later we write

$$\tilde{B}_n(f, x) := \sum_{\varrho_n \le k < an} + \sum_{an \le k \le n/2} := B_n^{(1)}(f, x) + B_n^{(2)}(f, x). \quad (6)$$

(In the case $\varrho_n > an$ the first sum is empty.) Furthermore let $\{0, 1, \ldots, n\} \setminus R_n = \{k_1 < k_2 < \cdots < k_m\}$. By the already quoted result of Golitschek [1], there exist c_j's such that

$$\left| y^k - \sum_{k < k_j \le n/2} c_j y^{k_j} \right| \le \prod_{k < k_j \le n/2} \frac{k_j - k}{k_j + k} \quad (7)$$

for every $0 \le y \le 1$. With these c_j's set

$$A_k(x) := x^k (1-x)^{n-k} - \sum_{k < k_j \le n/2} c_j x^{k_j} (1-x)^{n-k_j}$$

and

$$\tilde{A}_k(y) := y^k - \sum_{k < k_j \le n/2} c_j y^{k_j}.$$

Using (7), we have for $0 \le y \le 1$ and $k \in R_n$, $1 \le k \le an < n/4$

$$|\tilde{A}_k(y)| \le \prod_{k < k_j \le n/2} \frac{k_j - k}{k_j + k} \le \prod_{k + r_n \le j \le n/2} \frac{j - k}{j + k} = \prod_{k + r_n \le j \le n/2} \frac{1 - k/j}{1 + k/j}$$

$$\le \exp\left(-2k \sum_{k + r_n \le j \le n/2} \frac{1}{j}\right) \le \exp\left(-2k \int_{k + r_n}^{n/2} \frac{dx}{x}\right)$$

$$= \left(\frac{2r_n + 2k}{n}\right)^{2k}.$$

Now estimating $\tilde{A}_k(y)$ for $y > 1$, we use the well-known estimate for the Chebyshev polynomial of degree $[n/2]$ outside the interval $[0, 1]$:

$$|\tilde{A}_k(y)| \le \left(2y - 1 + 2\sqrt{y^2 - y}\right)^{n/2}\left(\frac{2r_n + 2k}{n}\right)^{2k} \qquad (y \ge 1).$$

By the last two estimates and the substitution $y = x/(1-x)$, we obtain

$$|A_k(x)| = \frac{|\tilde{A}_k(y)|}{(1+y)^n} \le \left(\frac{2r_n + 2k}{n}\right)^{2k} \max\left(1, \left[\frac{2y - 1 + 2\sqrt{y^2 - y}}{(1+y)^2}\right]^{n/2}\right)$$
$$= \left(\frac{2r_n + 2k}{n}\right)^{2k} \qquad (0 \le x \le 1).$$

Hence, there exists $B_n^*(x) \in \mathscr{P}(R_n)$ such that for $0 \le x \le 1$ (using Stirling's formula for estimating the binomial coefficients)

$$|B_n^{(1)}(x) - B_n^*(x)| \le \|f\| \sum_{\varrho_n \le k < an} \binom{n}{k} |A_k(x)|$$
$$\le \|f\| \sum_{\varrho_n \le k < an} \left(\frac{4e(k + r_n)^2}{kn}\right)^k$$
$$\le \|f\| \left[\sum_{\varrho_n \le k < r_n} \left(\frac{16er_n^2}{n\varrho_n}\right)^k + \sum_{r_n \le k < an} \left(\frac{16ek}{n}\right)^k\right]$$

(here, of course, we may have empty sums). Now if

$$a < \frac{1}{16e}, \tag{8}$$

then we can assure that the second sum in the last estimate tends to zero (we do not restrict generality in assuming that $r_n \to \infty$). In order for the first sum to converge to 0 as $n \to \infty$, it suffices that either

(a) $r_n = o(\sqrt{n})(\varrho_n \ge 1)$, or

(b) $\limsup_{n \to \infty} r_n^2/n\varrho_n < 1/16e$ and $\varrho_n \to \infty$.

Since $1/2^{15}e^2 < 1/16e$, under the assumptions of Theorems 1 and 2

$$\|B_n^{(1)}(x) - B_n^*(x)\| \to 0 \qquad (n \to \infty),$$

where $B_n^* \in \mathscr{P}(R_n)$. To complete the proof of sufficiency, it remains to approximate $B_n^{(2)}(f)$ by polynomials from $\mathscr{P}(R_n)$. Set

$$\gamma_{nk} := \min_{c_i} \max_{a^2 \le x \le 15/16} \left| 1 - \sum_{k < k_i < 3n/4} c_i x^{k_i - k}(1-x)^{k-k_i} \right|$$
$$= \min_{c_i} \max_{a^2/(1-a^2) \le y \le 15} \left| 1 - \sum_{k < k_i < 3n/4} c_i y^{k_i - k} \right|.$$

Furthermore, denote

$$p_{nk}(x) := f\left(\frac{k}{n}\right)\binom{n}{k} \sum_{k < k_i < 3n/4} c_i^* x^{k_i}(1-x)^{n-k_i} \in \mathscr{P}(R_n),$$

where the c_i^*'s are the solutions of the above extremal problem.

Let us estimate γ_{nk} using Lemma 1. Evidently, all the integers $k_i - k$ are between 1 and $3n/4 - k$, while their number m is $\ge 3n/4 - k - r_n$. In addition, both $r_n = o(\sqrt{n})$ and (3) imply that

$$r_n < \frac{n}{2^8 e} \le \left(\frac{3n}{4} - k\right)\frac{1}{2^6 e}$$

(since $1 \le k$, $\varrho_n \le n/2$). Thus $m \ge (3n/4 - k)(1 - \theta)$ with $\theta = 1/2^6 e$. Now apply Lemma 1 with this θ, n replaced by $3n/4 - k$ and

$$\theta_0 = \sqrt{\frac{a^2}{15(1-a^2)}} > \theta.$$

The latter inequality, as well as the previous condition (8) on a can be satisfied if a is close enough to $1/16e$.

We obtain that $\gamma_{nk} \to 0$ as $n \to \infty$ uniformly for every $1 \le k \le n/2$, i.e.,

$$\gamma_n := \max\{\gamma_{nk} : 1 \le k \le n/2\} \to 0 \qquad (n \to \infty). \tag{9}$$

Set now

$$D_n(x) := B_n^{(2)}(x) - B_n^{**}(x),$$

where

$$B_n^{**}(x) := \sum_{an \le k \le n/2} p_{nk}(x) \in \mathscr{P}(R_n).$$

Then for every $x \in [a^2, 15/16]$ we have

$$|D_n(x)| \le \|f\| \sum_{an \le k \le n/2} \binom{n}{k} x^k (1-x)^{n-k} \gamma_{nk} \le \gamma_n \|f\|. \qquad (10)$$

Note that $D_n(x)$ is a linear combination of polynomials $x^k(1-x)^{n-k}$ with $an \le k \le 3n/4$. Therefore D_n is a-incomplete at 0 and 1/4-incomplete at 1. Thus (see the remarks on incomplete polynomials in Section 2)

$$\|D_n\| = \max_{a^2 \le x \le 15/16} |D_n(x)|.$$

Hence and by (9)–(10), $\|D_n\| \to 0$ $(n \to \infty)$, i.e., $\|B_n^{(2)}(f) - B_n^{**}\| \to 0$ as $n \to \infty$. This completes the proof of sufficiency in Theorems 1 and 2.

For the proof of necessity we need an auxiliary result providing estimates for the coefficients c_k of a polynomial

$$p_n(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}. \qquad (11)$$

LEMMA 2. *Given a polynomial p_n of the form* (11) *we have*

$$|c_k| \le \binom{2n}{2k} \|p_n\| \qquad (0 \le k \le n). \qquad (12)$$

Proof. Let

$$T_n(x) = \sum_{k=0}^{n} d_{kn} x^k (x-1)^{n-k}, \qquad \|T_n\| = 1,$$

be the Chebyshev polynomial of degree n transformed to the interval [0, 1]. Then by Szegő [6, (4.3.2)],

$$d_{kn} = \binom{n-1/2}{k}\binom{n-1/2}{n-k} \Big/ \binom{n-1/2}{n} = \binom{2n}{2k} \qquad (k = 0, \ldots, n). \qquad (13)$$

We claim that

$$|c_k| \le d_{kn} \|p_n\| \qquad (k = 0, \ldots, n). \qquad (14)$$

If $|c_s| > d_{sn}\|p_n\|$ for some $0 \leq s \leq n$, then the polynomial

$$q_s(x) := \frac{T_n(x)}{d_{sn}} - \frac{p_n(x)}{c_s}$$

possesses n distinct zeros in the open interval $(0, 1)$. However,

$$q_s(x) = \sum_{\substack{j=0 \\ j \neq s}}^{n} a_j x^j (1-x)^{n-j} = (1-x)^n \sum_{\substack{j=0 \\ j \neq s}}^{n} a_j \left(\frac{x}{1-x}\right)^j$$

can have at most $n - 1$ zeros in $(0,1)$ since $\{t^j, 0 \leq j \leq n, j \neq s\}$ is an n-dimensional Chebyshev system on $(0, \infty)$. Thus (12) follows from (13)–(14).

Proof of Necessity in Theorems 1 *and* 2. For arbitrary integers r_n and ϱ_n such that $r_n + 2\varrho_n \leq n + 1$, set $R_n = \{\varrho_n, \varrho_n + 1, \dots, \varrho_n + r_n - 1\}$. Then (2) holds. For an arbitrary $p_n \in \mathscr{P}(R_n)$ we have

$$p_n(x) = \left\{ \sum_{k=0}^{\varrho_n - 1} + \sum_{k=\varrho_n + r_n}^{n} \right\} c_{kn} x^k (1-x)^{n-k} := p_{1,n}(x) + p_{2,n}(x). \tag{15}$$

Moreover, by (12), using again Stirling's formula

$$|p_{2,n}(x)| \leq \|p_n\| \sum_{k=\varrho_n + r_n}^{n} \left(\frac{en}{k}\right)^{2k} x^k \leq \|p_n\| \sum_{k=\varrho_n + r_n}^{n} \left(\frac{e^2 n^2 x}{r_n^2}\right)^k. \tag{16}$$

Assume now that $p_n(x) \to 1$ uniformly on $[0, 1]$. First let us consider the case when $\varrho_n = \varrho$ is fixed (Theorem 1), and assume that $r_n \geq \delta\sqrt{n}$ for a proper subsequence of integers n, with a $\delta > 0$. (In the rest of the proof we tacitly assume that n is an element of this, or similar subsequence.) Set $x = t\delta^2/2e^2 n$, $0 \leq t \leq 1$. Then

$$\begin{aligned} p_{1,n}(x) &= (1-x)^{n-\varrho+1} \sum_{k=0}^{\varrho-1} c_{kn} x^k (1-x)^{\varrho-k-1} \\ &= \left(1 - \frac{t\delta^2}{2e^2 n}\right)^{n-\varrho+1} q_n(t), \end{aligned}$$

where $q_n(t)$ is a polynomial of degree at most $\varrho - 1$. Furthermore, by (16),

$$\left|p_{2,n}\left(\frac{t\delta^2}{2e^2n}\right)\right| \le \|p_n\| \sum_{k=\varrho_n+r_n}^{n} \left(\frac{t}{2}\right)^k \to 0 \qquad \text{uniformly on } 0 \le t \le 1.$$

Hence, $p_{1,n}(t\delta^2/2e^2n) \to 1$ uniformly on $0 \le t \le 1$. However,

$$\left(1 - \frac{t\delta^2}{2e^2n}\right)^{n-\varrho+1} \to e^{-\alpha t} \qquad \left(\alpha = \frac{\delta^2}{2e^2}\right),$$

i.e., $q_n(t) \to e^{\alpha t}$ $(0 \le t \le 1)$, a contradiction. This verifies the necessary condition in Theorem 1.

Now let $r_n^2 \ge \beta n \varrho_n$, where $\beta > 53$. Using again (16), we obtain that

$$|p_{2,n}(x)| = o(1) \qquad \text{whenever} \quad 0 \le x \le \frac{cr_n^2}{n^2} := x_n, \tag{17}$$

with an arbitrary $0 < c < e^{-2}$. Therefore for sufficiently large n's

$$|p_{1,n}(x)| = (1-x)^{n-\varrho_n+1}|g_n(x)| \le 2 \qquad (0 \le x \le x_n), \tag{18}$$

where g_n is a polynomial of degree $\le \varrho_n - 1$. Thus

$$|g_n(x)| \le 2\left(1 - \frac{x_n}{2}\right)^{-n+\varrho_n-1} \qquad (0 \le x \le x_n/2).$$

Hence, using the growth properties of Chebyshev polynomials, we obtain

$$|g_n(x)| \le 2\left(1 - \frac{x_n}{2}\right)^{-n+\varrho_n-1}(3 + 2\sqrt{2})^{\varrho_n-1} \qquad (0 \le x \le x_n).$$

Thus by (18) and $\varrho_n \le n/2$

$$\begin{aligned}|p_{1,n}(x_n)| &\le 2\left(\frac{1-x_n}{1-x_n/2}\right)^{n-\varrho_n+1}(3+2\sqrt{2})^{\varrho_n-1} \\ &\le \left(1 - \frac{x_n}{2}\right)^{n-\varrho_n+1}(3+2\sqrt{2})^{\varrho_n} \le e^{(n/2)\log(1-x_n/2)}(3+2\sqrt{2})^{\varrho_n} \\ &\le e^{-cr_n^2/4n}(3+2\sqrt{2})^{\varrho_n} \le e^{(\log(3+2\sqrt{2})-c\beta/4)\varrho_n}.\end{aligned}$$

Since $\beta > 53$, when $c < e^{-2}$ is sufficiently close to e^{-2} we obtain that $c\beta/4 > \log(3 + 2\sqrt{2})$, i.e., $p_{1,n}(x_n) \to 0$ as $n \to \infty$. However, by (17) we also have $p_{2,n}(x_n) \to 0$, a contradiction. The proof of Theorems 1 and 2 is complete.

Note that in our proof of necessity of Theorems 1 and 2 we deal only with sets $R_n = \{\varrho_n, \varrho_{n+1}, \ldots, \varrho_n + r_n - 1\}$, since this structure of the set R_n gives a space $\mathscr{P}(R_n)$ with worst approximative properties. Thus formulating the necessity parts of Theorems 1 and 2 with these R_n's would lead only to a formally more general statement.

Theorems 1 and 2 provide asymptotically sharp conditions on $\# R_n$ which ensure (1). Of course, the question of exact constant in Theorem 2 remains open. The exact constant can be determined in the special case when R_n consists of consecutive integers from the "middle" of the sequence $\{1, \ldots, n-1\}$.

THEOREM 3. *Let*

$$R_n = \{k : [\alpha n] < k < [\beta n]\} \qquad (0 < \alpha < \beta < 1).$$

Then (1) *holds if* $(1-\alpha)^2 + \beta^2 < 1$, *and it fails to hold if* $(1-\alpha)^2 + \beta^2 > 1$.

Proof. It is easy to see that $\mathscr{P}(R_n)$ consists of sums $p + q$, where p and q are β- and $(1-\alpha)$-incomplete polynomials at 0 and at 1, respectively. Furthermore, any $f \in C[0,1]$ can be decomposed into $f = f_1 + f_2$, where $f_1, f_2 \in C[0,1]$, $f_1 \equiv 0$ on $[0, \beta^2]$ and $f_2 \equiv 0$ on $[1-(1-\alpha)^2, 1]$ (supposing $(1-\alpha)^2 + \beta^2 < 1$). It is known (cf. von Golitschek [1] and Saff and Varga [5]), that f_1 and f_2 can be uniformly approximated on $[0,1]$ by β- and $(1-\alpha)$-incomplete polynomials at 0 and at 1, respectively. Thus (1) holds.

Assume now that $(1-\alpha)^2 + \beta^2 > 1$ and let $t_n \in \mathscr{P}(R_n)$, $\|t_n\| \le 1$ be bounded polynomials. Then $t_n = p_n + q_n$, where p_n and q_n are β- and $(1-\alpha)$-incomplete polynomials at 0 and at 1, respectively.

Case 1: $\|p_n\| \le A$ $(n \in \mathbf{N})$. Then we also have $\|q_n\| \le A + 1$. Therefore $p_n(x) \to 0$ for $x \in [0, \beta^2)$ and $q_n(x) \to 0$ for $x \in (1-(1-\alpha)^2, 1]$, i.e., $t_n(x) \to 0$ on $(1-(1-\alpha)^2, \beta^2)$. Thus (1) cannot hold.

Case 2: $\limsup_{n\to\infty} \|p_n\| = \infty$. Then we also have $\limsup_{n\to\infty} \|q_n\| = \infty$. Since p_n is β-incomplete at 0, $p_n(x) = o(\|p_n\|)$ uniformly for $x \in [0, 1-(1-\alpha)^2] \subset [0, \beta^2]$. In addition, q_n being $(1-\alpha)$-incomplete at 1, it attains its norm on $[0, 1-(1-\alpha)^2]$, i.e., $\|q_n\| = 1 + o(\|p_n\|)$, a contradiction. Theorem 3 is proved.

3. The Space $C^*[0, 1]$

According to Theorems 1 and 2 the question of density of the polynomials $\mathscr{P}(R_n)$ is delicately related to the distance $\varrho(R_n)$ of the set R_n from the endpoints of the interval $[0, n]$. Therefore it is natural to expect that our problem will have a different solution for the space $C^*[0, 1]$. Of course, in this case we do not need to keep the first and last basis function $(1-x)^n$ and x^n. Thus we can choose any $R_n \subset \{0, \dots, n\}$ and ask whether

$$\operatorname*{Lim}_{n \to \infty} \mathscr{P}(R_n) = C^*[0, 1]. \tag{19}$$

If $\varrho(R_n) \geq cn$ with some $c > 0$, then by Theorem 2, (19) holds provided that $\# R_n \leq Mn$ (with a proper $M > 0$). Since this statement is asymptotically sharp, it remains to consider the situation when $\varrho(R_n) = o(n)$. Our next result shows that under this condition the density in $C^*[0, 1]$ holds in a much more general setting.

Theorem 4. *Let $\{r_n\}, \{\varrho_n\} (r_n + 2\varrho_n \leq n + 1)$ be increasing sequences of integers and assume $\varrho_n = o(n)$. Then in order that for every $R_n \subset \{0, \dots, n\}$ with (2) the relation (19) holds, it is necessary and sufficient that $r_n = o(n)$.*

Proof. The proof of sufficiency is essentially a simplified version of the proof of sufficiency in Theorem 2, so we give only an outline of it. Let $f \in C^*[0, 1]$ and choose an arbitrary $\varepsilon > 0$. Then there exists a $\delta > 0$ depending on ε and $f_\delta \in C^*[0, 1]$ such that $\|f - f_\delta\| \leq \varepsilon$ and $f_\delta \equiv 0$ on $[0, \delta] \cup [1 - \delta, 1]$. For a sufficiently large n we also have $\|f_\delta - B_n(f_\delta)\| \leq \varepsilon$, where

$$B_n(f_\delta, x) = \sum_{\delta n < k < (1-\delta)n} f_\delta\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Thus in representation (6) we need to consider only the term $B_n^{(2)}(f_\delta, x)$ (with δ instead of a). Then as in the proof of Theorem 2 we can approximate $B_n^{(2)}(f_\delta, x)$ by polynomials from $\mathscr{P}(R_n)$ provided that $r_n < \tilde{c}n$ with a proper $\tilde{c}$ depending on δ. Since $r_n = o(n)$, this relation will hold for sufficiently large n's.

In order to prove the necessity assume that $r_n > dn$ for some $d > 0$. Set $R_n = \{\varrho_n, \varrho_n + 1, \dots, \varrho_n + r_n - 1\}$. Then for an arbitrary $p_n \in \mathscr{P}(R_n)$ representation (15) holds. Therefore

$$p_n(x) = (1-x)^{n-\varrho_n} \tilde{p}_{1,n}(x) + x^{[dn]} \tilde{p}_{2,n}(x)$$
$$(\deg \tilde{p}_{1,n} \leq \varrho_n, \deg \tilde{p}_{2,n} \leq n - [dn]).$$

Assume that $\|p_n\| \leq 1$, $n \in \mathbf{N}$. Since $\varrho_n = o(n)$ for n sufficiently large we have $n - \varrho_n > bn$, where $1 > b^2 > 1 - d^2$. Now we can repeat the argument used in the proof of Theorem 3 and show that $p_n(x) \to 0$ for $x \in (1 - b^2, d^2)$. Thus (19) cannot hold, and Theorem 4 is proved.

Remark. A possible generalization of Theorem 3 is the case when

$$R_n = \{k: \alpha n \le k \le \beta n \text{ or } \gamma n \le k \le \delta n\} \qquad (0 < \alpha < \beta < \gamma < \delta < 1).$$

We could settle this by using the two-point incomplete polynomial result of He and Li [3].

4. Open Problems

We have already mentioned the question of narrowing the gap between conditions (3) and (4) in Theorem 2. Similarly, in Theorem 4 the case $(1 - \alpha)^2 + \beta^2 = 1$ is open, but this is easily seen to be equivalent to the study of behavior of θ-incomplete polynomials around the point θ^2, which is also unsolved (cf. Lorentz [4, p. 43]).

Our results above answer the question of density of $\mathscr{P}(R_n)$ in terms of $\#R_n$ and $\varrho(R_n)$. A more delicate (and difficult) problem consists in providing necessary and sufficient conditions for (1) to hold in case of an *arbitrary* sequence R_n.

Another interesting question is to give necessary and sufficient conditions for a sequence $\{n_k, m_k\} \in \mathbf{Z}_+^2$ so that $x^{n_k}(1 - x)^{m_k}$ $(k \ge 1)$ span $C^*[0, 1]$.

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